Dynamic excitations
of transport structures

Prof. Ing. Ladislav Frýba, DrSc., Dr.h.c.

Institute of Theoretical and Applied Mechanics, v.v.i.
Academy of Sciences of the Czech Republic, Prague
Outline

- Introduction
- Movement of loads
- Rails
- Random vibration of structures
- Degradation of structural materials under dynamic loads
- Conclusions
Introduction

Transport structures
- moving load
- earthquake
- wind

Movement of the load

Moving load

\[ f(x, t) = \delta[x - u(t)] \left\{ F - m \frac{d^2 v[u(t), t]}{dt^2} \right\} \]

\[ \frac{d^2 v[u(t), t]}{dt^2} = \frac{\partial^2 v}{\partial u^2} \left( \frac{du}{dt} \right)^2 + 2 \frac{\partial^2 v}{\partial u \partial t} \frac{du}{dt} + \frac{\partial v}{\partial u} \frac{d^2 u}{dt^2} + \frac{\partial^2 v}{\partial t^2} \]
Uniform movement at velocity $c$

\[ x = u(t) = ct, \quad \frac{du}{dt} = c, \quad \frac{d^2u}{dt^2} = 0 \]

\[ \frac{d^2v(ct, t)}{dt^2} = \left[ \frac{\partial^2 v(x, t)}{\partial t^2} + 2c \frac{\partial^2 v(x, t)}{\partial x \partial t} + c^2 \frac{\partial^2 v(x, t)}{\partial x^2} \right]_{x=ct} \]

Motion in $(x,y)$ plane $x = u(t), y = v(t)$

load $f(x,y,t) - m(x,y,t) \frac{d^2w}{dt^2}$

\[ \frac{d^2w}{dt^2} = \left[ \frac{\partial^2 w(du/dt)^2}{\partial x^2} + \frac{\partial^2 w(dv/dt)^2}{\partial y^2} + \frac{\partial^2 w}{\partial t^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{du}{dt} \frac{dv}{dt} + \right. \]

\[ \left. +2 \frac{\partial^2 w}{\partial x \partial t} \frac{du}{dt} + 2 \frac{\partial^2 w}{\partial y \partial t} \frac{dv}{dt} + \frac{\partial w}{\partial x} \frac{d^2u}{dt^2} + \frac{\partial w}{\partial y} \frac{d^2v}{dt^2} \right]_{x=u(t), \quad y=v(t)} \]
Differential equation

\[ EIv^{IV}(x,t) + \mu \ddot{v}(x,t) + 2\mu \omega_b \dot{v}(x,t) + kv(x,t) = \delta(x-ct)P \]

Steady state vibration

\[ v(x,t) = v_0 v(s), \quad v_0 = \frac{P}{8\lambda^3EI} = \frac{P\lambda}{2k} \]

\[ \lambda = \left( \frac{k}{4EI} \right)^{1/4} \]
Moving coordinate: \[ s = \lambda (x - ct) \]

Speed parameter: \[ \alpha = \frac{c}{c_{cr}} = \frac{c}{2\lambda} \left( \frac{\mu}{EI} \right)^{1/2} \]

Damping parameter: \[ \beta = \omega_b (\mu / k)^{1/2} \]

Critical speed: \[ c_{cr} = 2\lambda \left( \frac{EI}{\mu} \right)^{1/2} \]

Ordinary differential equation:
\[
\nu^{IV} (s) + 4\alpha^2 \nu'' (s) - 8\alpha\beta \nu' (s) + 4\nu(s) = 8\delta(s)
\]
Solution

\[ v(s) = \frac{2}{a_1(D_1^2 + D_2^2)} e^{-bs} (D_1 \cos a_1 s + D_2 \sin a_1 s) \]

for \( s > 0 \)

\[ v(s) = \frac{2}{a_2(D_3^2 + D_4^2)} e^{bs} (D_3 \cos a_2 s + D_4 \sin a_2 s) \]

for \( s < 0 \)
Random vibration of a beam on elastic foundation under moving force
Equation of the beam

\[ L[v(x, t)] = EIv^{IV}(x, t) + \mu\ddot{v}(x, t) + 2\mu\omega_0\dot{v}(x, t) + k(x)v(x, t) = \delta(x - ct) \cdot F(t) \]

Foundation

\[ k(x) = k + \varepsilon k(x) \]
\[ k = E[k(x)] \]

Force

\[ F(t) = F + \varepsilon F(t) \]
\[ F = E[F(t)] \]

Moving coordinate

\[ s = \lambda(x - ct) \quad , \quad \lambda = \left(\frac{k}{4EI}\right)^{1/4} \]

Static deflection and bending moment under force \( F \)

\[ v_0 = \frac{F}{8\lambda^3 EI} = \frac{F\lambda}{2k} \quad , \quad M_0 = \frac{F}{4\lambda} \]
Coefficient of variation $V_V$ and $V_M$ as a function of speed $\alpha$, $\beta = 0.2$, $\gamma_i = 0$, $\sigma = 0.2$, $D = 10/\lambda$
COVARIANCE

(a)

(b)

(c)

(d)

(e)

POWER SPECTRAL DENSITY
Random vibration of a beam

Bernoulli-Euler equation

\[ EI \, v^{IV}(x,t) + \mu \ddot{v}(x,t) + 2\mu \omega_b \dot{v}(x,t) = p(x,t) \]

Normal mode analysis

\[ v(x,t) = \sum_{j=1}^{\infty} v_j(x) \, q_j(t) \]
\[ p(x,t) = \sum_{j=1}^{\infty} \mu \, v_j(x) \, Q_j(t) \]

Generalized deflection

\[ \ddot{q}_j(t) + 2\omega_b \dot{q}_j(t) + \omega_j^2 \, q_j(t) = Q_j(t) \]

Generalized force

\[ Q_j(t) = \begin{cases} \frac{1}{M_j} \int_0^L p(x,t) \, v_j(x) \, dx & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \]
Solution

\[ q_j(t) = \int_{-\infty}^{\infty} h_j(t - \tau) Q_j(\tau) d\tau = \int_{-\infty}^{\infty} h_j(\tau) Q_j(t - \tau) d\tau \]

Impulse function (weighting function)

\[ h_j(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_j(\omega) e^{i\tau \omega} d\omega \]

Frequency response function

\[ H_j(\omega) = \int_{-\infty}^{\infty} h_j(\tau) e^{-i\omega \tau} d\tau \]

Load

\[ p(x,t) = E[p(x,t)]' + p(x,t) \]
Stationary vibration of a beam under moving continuous random load
Load

\[ p(x,t) = \left( p + \varepsilon \, p(s) \right) \cdot \left[ 1 + \varepsilon \, r(t) \right] \]

\[ p = E[p(x,t)] \]

\[ p(x,t) = \varepsilon \, p(s) + \varepsilon \, p \, r(t) + \varepsilon^2 \, p(s) \, r(t) \]

\[ s = t - x / c \]

\[ \varepsilon \leq 1 \]

Covariance of the moving load

\[ C_{pp}(x_1, x_2, \tau) = \varepsilon^2 C_{pp}(\tau + \tau_0) + \varepsilon^2 \rho^2 C_{rr}(\tau) + \varepsilon^2 pC_{pr}(\tau + x_1 / c) + \]

\[ + \varepsilon^2 pC_{rp}(\tau - x_2 / c) + \varepsilon^3 C_{ppr} + \varepsilon^3 pC_{prr} + \varepsilon^3 C_{ppr} + \varepsilon^3 pC_{tpp} + \varepsilon^4 C_{pprr} \]

\[ \tau_0 = \frac{x_1 - x_2}{c} \]
Non-stationary vibration of a beam under moving random force
Load
\[ p(x,t) = \delta(x - ct) \ P(t) \, , \quad P(t) = P + \dot{P}(t) \, , \quad E[P(t)] = P \]

Covariance of the force
\[ C_{pp}(t_1, t_2) = E\left[ \ddot{P}(t_1) \ddot{P}(t_2) \right] \]

Covariance of moving load
\[ C_{pp}(x_1, x_2, t_1, t_2) = \delta(x_1 - ct_1) \delta(x_2 - ct_2) C_{pp}(t_1, t_2) \]

Covariance of generalized deflection
\[ C_{qjqk}(t_1, t_2) = \frac{1}{M_j M_k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_j(t_1 - \tau_1) h_k(t_2 - \tau_2) v_j(c \tau_1) v_k(c \tau_2) C_{pp}(\tau_1, \tau_2) d\tau_1 d\tau_2 \]

Speed parameter \[ \alpha = c/(2 f_1 l) \]

Damping parameter \[ \beta = \omega_b / \omega_1 = \vartheta / (2\pi) \]

Simple beam \[ v_j(x) = \sin \frac{j \pi x}{l} , \quad v_0 = \frac{P l^3}{48 E I} \]
Degradation of structural materials under dynamic loads

Dynamic loads

Wöhler curve

Fatigue tests
Concept of fatigue assessment

- Static system
- Material properties (Wöhler curve)

- "Real" Loads \( P(t) \)
- Stress curve \( \sigma(t) \)
- Spectrum \( n_i(\sigma_o, \sigma_n) \)
  - Calculation value for damage \( D_{sd} \)
    - Proof \( D_{sd} \leq D_{lim} \)

Simulation of train passages
- Counting procedure (Rainflow)
- Damage hypothesis (Palmgren-Miner)

\[
\Delta \sigma = \sigma_{\text{max}} - \sigma_{\text{min}} \\
\rho = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}
\]
Strategy of maintenance, repairs and reconstructions,

yes – the conditions are fulfilled,
not – the conditions are not fulfilled
Conclusions

- The dynamic loads, i.e. in time varying loads, increase the stresses in transport structures with increasing speed.
- The dynamic loads are varying in time either regularly or randomly.
- The response of structural materials deteriorates their properties in time.